such that the parametric equation for the second arc becomes

$$x_i^2(t) = \sum_{j=1}^{4} R_{ij}(t)x_j^1(t_1) + \alpha\int_0^t [R_{i2}(t-\tau)\cos\beta(\tau) + R_{i4}(t-\tau)\sin\beta(\tau)]\mathrm{d}\tau \tag{31}$$

On the third arc, we have

$$x_5 = \Omega_2 \tag{32}$$

hence,

$$x_5^3(t) = \Omega_2 t + C_5^3 \tag{33}$$

If we impose the conditions at the end of the second arc, the constant of integration

$$C_5^3 = \beta(t_2) - \Omega_2 t_2 \tag{34}$$

is determined. It follows that

$$x_5^3(t) = \Omega_2 t + \beta(t_2) - \Omega_2 t_2 = \gamma(t) \tag{35}$$

Similarly, with (31) we obtain

$$x_6^3(t) = \sum_{j=1}^{4} R_{ij}(t)x_j^2(t_2) + \alpha\int_{t_2}^t [R_{i2}(t-\tau)\cos\gamma(\tau) + R_{ij}(t-\tau)\sin\gamma(\tau)]\,\mathrm{d}\tau \tag{36}$$

Imposing the final conditions yields

$$\sum_{j=4}^{4} R_{2j}(T)x_j^2(t_2) + \alpha\int_{t_2}^T [R_{22}(t-\tau)\cos\gamma(\tau) + R_{24}(t-\tau)\sin\gamma(\tau)]\,\mathrm{d}\tau = c \tag{37a}$$

$$\sum_{j=1}^{4} R_{4j}(T)x_j^5(t_2) + \alpha\int_{t_2}^T [R_{42}(t-\tau)\cos\gamma(\tau) + R_{44}(t-\tau)\sin\gamma(\tau)]\,\mathrm{d}\tau = d \tag{37b}$$

$$\Omega_2 T - \beta(t_2) - \Omega_2 t_2 = \varphi \tag{37c}$$

The transition from the second interval to the third one does not require additional conditions because the integration constants are determined from the relations

$$\psi_5(t_2) = 0 \tag{38a}$$

$$\mathrm{tg}x_5(t_2) = \frac{\psi_4(t_2)}{\psi_2(t_2)} \tag{38b}$$

Making explicit (19), (21), and (22), we will obtain

$$\psi_1^0\left[\beta_1 e^{at1}\sin\Omega_1 t_1 + \beta_2 e^{at1}\cos\Omega_1 t_1 - \beta_2 - \frac{\alpha}{\Omega_1}(-a^2 + K_2)\right] + \psi_2^0\left[-\beta_1 e^{-at1}\sin\Omega_1 t_1 + \beta_2 e^{-at_1}\cos\Omega_1 t_1 - \beta_2 - \frac{\alpha}{\Omega_1}(-a^2 + K_2)\right] + \psi_3^0\left[\beta_3\cos(\ell - \Omega_1)t_1 + \beta_u\cos(\ell + \Omega_1)t_1 - \beta_3 - \beta_4 - \frac{\alpha}{\Omega_1}(\ell^2 + \Omega_1^2)\right] + \psi_4^0\left[\beta_3\sin(\ell - \Omega_1)t_1 + \beta_4\sin(\ell + \Omega_1)t_1\right] = 0 \tag{39a}$$

$$\psi_1^0 a(4\omega^2 - K_2 + a^2)e^{aT} - \psi_2^0 a(4\omega^2 - K_2 + a^2)e^{-aT} - \psi_3^0\ell \times (4\omega^2 - K_2 - \ell^2)\sin\ell T + \psi_4^0\ell(4\omega^2 - K_2 - \ell^2)\cos\ell T = 0 \tag{39b}$$

$$\psi_1^0 e^{aT} + \psi_2^{0-aT} + \psi_3^0\cos\ell T + \psi_4^0\sin\ell T = 0 \tag{39c}$$

The relations (39) form an algebraic system with the unknowns $\psi_K^0/\psi_1^0(K = 2, 3, 4)$ whose function expressions of t_1 and T are used in the developments that give the state variables on the second and third interval of time.

In order to calculate t_1, t_2, and T, we have to take into account the fulfillment of the condition $0 \le t_1 \le t_2 \le T$. The unknowns are the five integration constants that come from the adjoint system and t_1, t_2, and T. In order to determine the eight unknowns, relations (37) and (39) are used. The determination of the time of optimal transfer will be done by considering all variants (20), among with the one that for T gives the smallest positive value is picked up. In this way, we obtain the succession of arcs of the optimal trajectory.

The motions with $u = \pm\Omega$, as well as those with $\mathrm{tg}x_5 = \psi_2(t)/\psi_4(t)$, are optimal. The foregoing analysis shows that the optimal trajectory consists of three arcs, among which those situated at the extremities are determined by the control function $u = \pm\Omega$. The intermediate portion corresponds to the tangent law. We have to note that a complete treatment that would permit one to calculate the time of the arcs of the optimal trajectory is possible only by numerical integration. For the simplified equations of motion of a material point,[1] the tangent law gives a constant value for the direction of the propulsion. Our analysis shows that the motion around the collinear libration pointe gives a variation in terms of time of the orientation angle of the propulsion that considerably complicates the problem. Thus, the determination of the unknowns implies the solution of some integral equations and of nonlinear algebraic systems.

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Minimum Impulse Orbital Evasive Maneuvers

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Introduction

MUCH research has been done in the dynamics of orbital transfer and rendezvous, but few papers have dealt with the problem of avoiding an interception in space. Many military capabilities are being put in satellites and the means of intercepting them are improving. Some research has been done

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on optimal interception by Prussing and Heckathorn.[1] The problem of avoiding an interception also needs attention. One way to defeat an interception is to maneuver the attacked satellite so that it does not come within the lethal radius of the attacker. However, any orbital maneuver will use up the satellite's supply of propellant, shortening its useful life or leaving it vulnerable to a second attack. All other things being equal, the evasive maneuver should be as small as possible. The research described in this Note is directed at this problem.

An algorithm was developed that calculated the smallest possible impulsive evasive maneuver under the assumption of Keplerian dynamics when given a satellite-state vector, a threat-state vector, an avoidance radius, and a time at which to perform an evasive maneuver. This Note describes the results of the algorithm when applied to a realistic scenario for the orbital interception of a geosynchronous satellite. Results are also included that show how the size of the evasive maneuver changed as the attacker's orbital parameters changed. All of these results are presented in greater detail by Burk.[2]

Mathematical Model

Let us assume that a satellite in a Keplerian orbit is subjected to a threat that can be described as another body in a Keplerian orbit and a lethal distance around it. The distance might represent the lethal radius of a warhead or the range of a radar. At a given time the satellite will be within the sphere defined by the threat's location and the lethal distance around it. A maneuver time is specified. The problem is to find the smallest velocity change that will result in the satellite staying outside the threat sphere. This is illustrated in Fig. 1. If the satellite does not maneuver, it will pierce the threat sphere at B and arrive at C when the threat arrives at A. Two evasive trajectories, DE and DF, are shown based on impulsive maneuvers at D. The sphere is moving with the attacker, and so the optimal evasion trajectory should be tangent to the sphere in the attacker's frame of reference. Also, time of flight is a variable since the optimal closest approach will not necessarily be at the same time as the nominal interception.

Method of Solution

The method of differential corrections developed by Escobal[3] was used to find the small-value approximation for the relationships between changes in velocity at the time of maneuver and changes in time of flight and in position and velocity at the time of interception. An iterative algorithm based on these corrections and incorporating the tangency constraint converged on a locally optimal maneuver. To ensure that the globally optimal solution was found, each problem was run nine times with different initial guesses. No more than two locally optimal solutions were ever found. No fewer than two were found unless the original interception came no closer to the target than about one-third of the threat radius. Two locally optimal solutions are probably the most that ever occur.

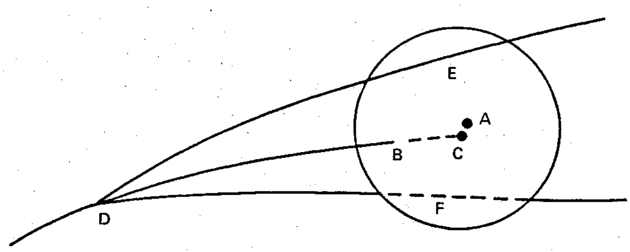

Fig. 1 Threat sphere and evasive trajectories.

Optimum Evasive Maneuver for a Standard Interception

A satellite interception scenario to use as standard or typical was selected in the following way. A satellite in a geosynchronous orbit was chosen as the target of the attack. An attack by direct ascent from the surface of the Earth was chosen as a likely near-term threat. The threat was given an orbit with apogee at some nominal intercept (collision) point. A launch site in the middle northern latitudes would be plausible, and so an orbital inclination of 45 deg was chosen. An apogee velocity of 1 km/s was selected as a round number that left perigee within the Earth. A threat radius of 300 km was picked. A maneuver interval of 7000 s was selected because it was about a quarter of the attacker's orbital period. The orbits of the target satellite and attacker were referenced to an equatorial x-y plane with $+z$ in the direction of north. The nominal intercept occurred as the target crossed the $+x$ axis in the $+y$ direction. At intercept the attacker had equal velocity components in the $+y$ and $-z$ directions.

The two locally optimal evasive maneuvers for this situation are shown in Table 1. Solution 1 is the global optimum with a maneuver impulse of 39.6480 m/s. Solution 2 is a maneuver generally opposite to the global optimum. It is noteworthy that the nonzero z components show that some degree of out-of-plane maneuvering can be optimal for noncoplanar intercept geometries.

Variations on the Standard

In order to investigate the sensitivity of maneuver size to the parameters of the attack, eight parameters of the standard interception were varied. Table 2 lists the parameters and shows the limits of variation of each. Table 3 contains the largest and the smallest maneuvers that were found as each parameter was varied. The lethal radius of the threat and the time of the maneuver had the strongest influence. Impulse size

Table 1 Locally optimal maneuvers for a standard interception

	Solution 1	Solution 2
Impulse magnitude, m/s	39.6480	39.7024
x-component, m/s	39.4977	−39.5928
y-component, m/s	−2.5889	1.2375
z-component, m/s	2.2791	−2.6762
Closest approach time (seconds from nominal intercept)	5.895	−6.961
Relative position of satellite from attacker at closest approach, km		
x component	299.12	−299.13
y component	−12.13	0.50
z component	19.46	−22.83

Table 2 Variations made on the standard interception

Parameter varied mnemonic units	Low value	Increment	High value
Sphere radius/SR/km	50	50	1000
Maneuver time before intercept/TM/s	500	500	80000
Attacker speed at intercept/AS, km/s	0.25	0.25	5
Attacker orbit inclination/AI, deg	0	15	360
Attacker flight path angle/AF, deg	0	15	360
Radial miss distance/RM, km	−250	50	+250
In-track miss distance/IM, km	−250	50	+250
Cross-track miss distance/CM, km	−250	50	+250

Table 3 Extremes of variations on the standard interception

P	PL	ML	PH	MH	%
SR	50	6.612	1000	132.010	1897
TM	80	3.932	500	599.756	15153
AS	4.75	39.398	0.25	39.674	0.70
AI	60/300	39.646	180	39.685	0.10
AF	315	39.398	75	40.723	3.36
RM	−250	6.615	0	39.648	499
IM	250	33.503	0	39.648	18
CM	250	8.981	0	39.648	341

[a]Key: P = parameter varied (mnemonics correspond to Table 2), PL/PH = parameter value that resulted in the smallest/largest maneuver (units as in Table 2), ML/MH = size of maneuver in m/s corresponding to PL/PH, % = difference between ML and MH expressed as a percentage of ML.

increased linearly as threat radius increased, and increased approximately exponentially as maneuver interval shrank. One surprising result was the small effect of variations in the interceptor's velocity, which altered the size of the evasive maneuver by only a few percent. Errors in the radial and cross-track directions resulted in nearly linear reductions in the size of the required maneuver. Errors in in-track miss distance produced a more gradual drop-off in maneuver size.

Summary and Suggestions for Further Work

A threat sphere moving in a Keplerian orbit was used to model an attack on a satellite. Defeat of the attack was considered by changing the satellite's orbital velocity at a specified time to avoid penetration of the threat sphere. An algorithm using a method of differential corrections was developed to find the minimum-impulse evasive maneuver. A 45 deg inclination impulsive direct ascent attack on a geosynchronous satellite with a threat sphere radius of 300 km and a maneuver time about 2 h before the nominal impact time was used as a reference case. The optimal evasive maneuver had a velocity change of 39.648 m/s with a nonzero out-of-plane component. The sensitivity of this solution to variations in intercept parameters was also studied. Maneuver time was the most important parameter, followed by threat sphere radius. In every case no more than two locally optimal solutions were found.

Further study of orbital evasive maneuvers along the lines drawn here might include the effects of uncertainty, improvement of accuracy of the model, and consideration of methods of jointly optimizing evasive maneuver size and other relevant factors. The algorithm used here takes no account of uncertainties in the data, either in knowledge of the state vectors or in the precision with which a recommended maneuver can be made. In reality these uncertainties might be significant. In the area of improving the fidelity of the model there are three interesting possibilities: non-Keplerian dynamics, nonimpulsive maneuvers, and nonspherical threat volumes (e.g., the acquisition cone of a radar). Perhaps the most important area of study is the optimization of other things along with impulse size. Such things as predictability, required attitude changes, and the return to the mission orbit might be significant factors in selecting an evasive maneuver. The best maneuver in a given situation will be determined by considering all of these factors.

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Method for Stability Analysis of an Asymmetric Dual-Spin Spacecraft

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Introduction

A DUAL-SPIN spacecraft may be approximately modeled as a gyrostat. Leimanis,[1] Kane,[2] Cochran,[3] and Tsuchiya[4] have each provided the analytical solution or nutational stability for the attitude motion of an axisymmetric or asymmetric gyrostat by use of Euler's equations. In this paper, first-order differential equations with variables, the Euler angles, and the angle of relative rotation are derived instead of Euler's equations.[5] From the first-order equations, the nutational stability and the criteria for stable quasipermanent spin are directly determined. The behavior of unstable attitude motion is investigated with the aid of the energy integral of the system.

First-Order Equations of Motion

The dual-spin spacecraft S consists of rigid bodies A and B connected by a bearing axis (see Fig. 1). A^*, B^*, and S^*, the respective mass centers of A, B, and S, lie on the bearing axis. The distance between A^* and B^* is 1. Body A is an asymmetric rotor. Its centroidal principal axis system is $(A^* - X_1X_2X_3)$. Body B is an axisymmetric platform. Its axis of symmetry is aligned with the X_1-axis. The $(B^* - X_1Y_2Y_3)$ system is fixed in body B, the orientation of which relative to $(B^* - X_1X_2X_3)$ is specified by the angle α. Bodies A and B and system S have masses m_A, m_B, and m, respectively. Their individual centroidal inertia dyadics $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{I}$ are

$$\boldsymbol{A} = \sum_{i=1}^{3} A_i\boldsymbol{x}_i\boldsymbol{x}_i, \quad \boldsymbol{B} = \sum_{i=1}^{3} B_i\boldsymbol{x}_i\boldsymbol{x}_i, \quad \boldsymbol{I} = \sum_{i=1}^{3} I_i\boldsymbol{x}_i\boldsymbol{x}_i \tag{1}$$

$$I_1 = A_1 + B_1, \quad I_i = A_i + B_3 + l^2m_Am_B/m \ (i = 2, 3) \tag{2}$$

where $\boldsymbol{x}_i$ are unit vectors parallel with axis X_i $(i = 1, 2, 3)$.

Establish $(S^* - Z_1Z_2Z_3)$, the coordinate system of angular momentum, with Z_1-axis along the angular momentum vector of system S about S^*. Let ψ, ϑ and φ be Euler angles of the centroidal principal axis system $(S^* - X_1X_2X_3)$ relative to $(S^* - Z_1Z_2Z_3)$ (see Fig. 2). Since there is no external torque, the rotational angular momentum of the system is constant. It may be expressed in the form

$$\boldsymbol{H} = Hc\vartheta\boldsymbol{x}_1 + Hs\vartheta s\varphi\boldsymbol{x}_2 + Hs\vartheta c\varphi\boldsymbol{x}_3 \tag{3}$$

Here H is the magnitude of $\boldsymbol{H}$ and is a first integral; $c(\ )$ and $s(\ )$ are defined as $\cos(\ )$ and $\sin(\ )$, respectively, for convenience. According to its definition, $\boldsymbol{H}$ may also be written as

$$\boldsymbol{H} = \boldsymbol{I}\cdot\boldsymbol{\omega} + \boldsymbol{B}\cdot\boldsymbol{\Omega} \tag{4}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ are the angular velocity of B relative to A and of A, respectively, which may be expressed as

$$\boldsymbol{\Omega} = \Omega\boldsymbol{x}_1 \tag{5a}$$

$$\boldsymbol{\omega} = \sum_{i=1}^{3} \omega_i\boldsymbol{x}_i \tag{5b}$$

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